Published by Institute of Physics Publishing for SISSA/ISAS

Received: *October 7, 2004*
Revised: *January 31, 2005*
Accepted: *February 1, 2005*

On Ricci flat supermanifolds

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Abstract: We study the Ricci flatness condition on generic supermanifolds. It has been found recently that when the fermionic complex dimension of the supermanifold is one the vanishing of the super-Ricci curvature implies the bosonic submanifold has vanishing scalar curvature. We prove that this phenomena is only restricted to fermionic complex dimension one. Further we conjecture that Calabi-Yau theorem holds for supermanifolds whose complex fermionic dimension is larger than one.

Keywords: Sigma Models, Differential and Algebraic Geometry.

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1. Introduction

Calabi-Yau compactification has been one of the most important cornerstones of superstring phenomenology. It is a supersymmetric vacuum solution to string theory in the absence of RR and NS-NS fluxes. Much progress has been made in studying sigma models, topological models and branes on Calabi-Yau backgrounds.

The extension to Calabi-Yau supermanifold had been attempted several years ago [1, 2], and received much attention recently after Witten's proposal that the perturbative amplitudes of the $\mathcal{N}=4$ super Yang-Mills theory can be recovered from open string theory on the Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$ [3]. It belongs to a class of supermanifolds which can be obtained starting from a certain bosonic vector bundle over a Kähler manifold and then fermionizing the bundle direction. The global holomorphic top form exists as long as the base manifold and the vector bundle have the same canonical line bundle. It is reasonable to compactify the string theory on supermanifolds and look for conformal backgrounds.

By the famous Calabi-Yau theorem, for given complex structure and Kähler class on a Kähler manifold, there exists a unique Ricci flat metric if and only if the first Chern class of the manifold vanishes, or there is a globally defined holomorphic top form on the manifold. Since the worldsheet sigma model is conformal invariant only when the target space is Ricci flat, it follows that the above-mentioned class of supermanifolds are all valid perturbative string theory backgrounds.

It is then a surprise that Rǒcek and Wadhwa [4] found a counterexample to the Calabi-Yau theorem when the supermanifolds are constructed by fermionizing a line bundle over the base manifold. They proved that in this case, the super-Ricci flatness actually requires more than just the vanishing of the first Chern class: it also requires the bosonic base manifold to have vanishing scalar curvature. The novelty of the supermanifold compared to the bosonic manifold, regarding the Ricci flatness, is that in addition to the vanishing of first Chern class as an integrability condition, there are local constraints from the fermionic expansion of the curvature, as we will analyze later.

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We will retain the name Calabi-Yau manifolds for Kähler manifolds with vanishing first Chern class, or equivalently with a global holomorphic top form, in the case of supermanifolds. So by the result of [4], $\mathbb{CP}^{3|1}$ is a super Calabi-Yau manifold, as it has a global holomorphic $(3, 1)$ form, but it is not super Ricci-flat as the base manifold $\mathbb{CP}^3$ has non-vanishing scalar curvature.

A natural question is whether this counterexample to the Calabi-Yau theorem is merely an exception restricted to fermionic complex dimension one, or more general valid for higher fermionic dimension. We will show in this paper that for the Ricci-flat metric on supermanifold to imply that the bosonic manifold has vanishing scalar curvature, the condition of the complex fermionic dimension being one is not only sufficient but also necessary. An intuitive explanation can be seen in the following simple example. Consider a function of z_0 and z_i. The Taylor expansion in z_0 has infinitely many terms

$$F(z_0, z_i) = f(z_i) + f_1(z_i)z_0 + f_1(z_i)z_0^2 + \cdots . \tag{1.1}$$

However, after fermionizing z_0, the expansion will be cut off in the second order due to the grassmannian nature of the fermionic coordinates,

$$F(\theta, z_i) = f(z_i) + f_1(z_i)\theta . \tag{1.2}$$

The natural disappearance of the higher order terms, as will be shown in this paper, directly leads to the result of [4].

There are additional local constraints for the various Taylor expansion coefficients of the supermetric from the Ricci-flatness. Although it is hard to analyze these conditions, we conjecture that they do not impose additional global topological constraints on the base manifold, but merely give the relations among the coefficients of the fermionic expansion of the supermetric.

2. Ricci-flat Bosonic Kähler manifold and superextension

To understand the peculiarity of the Calabi-Yau supermanifolds, let us first review the content of the usual Calabi-Yau theorem. It concerns the existence of the metric of SU(N) holonomy on a general Kähler manifold, which says that the necessary and sufficient condition for its existence is that the manifold has vanishing first Chern class. There are equivalent conditions which we will find useful later. For example, a compact Kähler manifold has vanishing first Chern class if and only if the manifold admits a nowhere vanishing holomorphic top form. Also the metric of SU(N) holonomy and the Ricci-flat metric are the same thing, see, for example, p. 439 of [5]. A Kähler manifold has the nice property that its Ricci curvature tensor is simply related to the Kähler metric as

$$R_{i\bar{j}} = -(\ln \det(g))_{,i\bar{j}} . \tag{2.1}$$

So locally, $R_{i\bar{j}} = 0$ implies

$$\ln \det(g) = F(z^i) + \bar{F}(\bar{z}^{\bar{i}}) , \tag{2.2}$$

for an arbitrary holomorphic function $F(z^i)$. By an appropriate holomorphic change of coordinates, $F(z)$ can be put into any form, for example, a constant. So then locally the Ricci-flatness condition implies that the Kähler potential obeys:

$$\det(g) \equiv \det(\partial_i \partial_{\bar{j}} K) = 1\,, \tag{2.3}$$

where K is the Kähler potential. This is a differential equation about a scalar function K, or an algebraic equation about the metric tensor, whose solution always exists. However this is just the local condition, and one has to patch all the local solutions together into a global one. The global integrability condition for the Ricci-flatness is then the vanishing of first Chern class.

However, in the case of the supermanifolds, this local equation implies additional constraints on the structure of the manifold, besides the global topological condition. To explain this point, and to compare to the case of the higher fermionic dimension, let us study more carefully the case of the fermionic complex dimension one. We will follow the derivation of [4] in this case.

In the following we will assume the block form of the supermetric

$$\mathcal{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2.4}$$

where A and D are bosonic, but C and D are fermionic. The superdeterminant is defined as

$$\mathrm{sdet}(\mathcal{G}) \equiv \frac{\det \mathrm{A}}{\det(D - CA^{-1}B)} = \frac{\det(A - BD^{-1}C)}{\det D}\,. \tag{2.5}$$

Now for supermanifolds with only one complex fermionic dimension, the fermionic expansion of the Kähler potential is

$$K = G + F\theta\bar{\theta}, \tag{2.6}$$

and the expansion of the supermetric follows

$$\mathcal{G} = \begin{pmatrix} G_{,i\bar{j}} + F_{,i\bar{j}}\theta\bar{\theta} & F_{,i}\theta \\ F_{,\bar{j}}\bar{\theta} & F \end{pmatrix}. \tag{2.7}$$

One should regard $G_{,i\bar{j}}$ as the Kähler metric on the bosonic submanifold, which only depends on z^i and $\bar{z}^{\bar{i}}$. Now a simple calculation yields

$$\begin{aligned} \mathrm{sdet}(\mathcal{G}) &= \frac{1}{F} \det\left(G_{,i\bar{j}} + F_{,i\bar{j}}\theta\bar{\theta} - \frac{F_{,\bar{j}}F_{,i}}{F}\theta\bar{\theta} \right) \\ &= \frac{\det(G_{,i\bar{j}})}{F} \left[1 + (G_{,i\bar{j}})^{-1} \left(F_{,i\bar{j}} - \frac{F_{,\bar{j}}F_{,i}}{F} \right) \theta\bar{\theta} \right]. \end{aligned} \tag{2.8}$$

Then $\mathrm{sdet}(\mathcal{G}) = 1$ implies

$$F = \det(G_{,i\bar{j}}), \tag{2.9}$$

$$0 = (G_{,i\bar{j}})^{-1}\left(F_{,i\bar{j}} - \frac{F_{,\bar{j}}F_{,i}}{F}\right), \tag{2.10}$$

and, using the first equation, one immediately realizes that the second equation means exactly the scalar curvature of the bosonic manifold vanishes

$$R \equiv (G_{,i\bar{j}})^{-1}R_{i\bar{j}}(\text{bosonic}) = (G_{,i\bar{j}})^{-1}[\ln\det(G_{,i\bar{j}})]_{,i\bar{j}} = (G_{,i\bar{j}})^{-1}(\ln F)_{,i\bar{j}} = 0\,. \tag{2.11}$$

Because of the Grassmann nature of the fermionic coordinates, both the Kähler potential and the super metric has a cutoff in the fermionic expansion. Were these fermionic coordinates bosonic and one does the same expansion, there would have been an infinite number of equations for an infinite number of coefficients of the expansion. One can then interprets them as determining the higher order expansion coefficients in terms of the lower order ones, which essentially encodes the trivial local equation that we mentioned above. However, in the fermionic case, the Grassmann nature imposes an a priori condition that infinitely many coefficients have to vanish. In turn they imposes consistency conditions on the lower order expansion coefficients if one still require it to be the solution. They will appear as additional constraints on the bosonic submanifold, as we have clearly seen from the example of the complex fermionic dimension one.

This leads to the conclusion of [4] that the Calabi-Yau theorem does not hold for the supermanifolds with one fermionic complex dimension one. In this case, the Ricci-flat supermetric requires not only the vanishing of the first Chern class, or equivalent the existence of the global holomorphic top form, but also requires that the base manifold has vanishing scalar curvature. For example, $\mathbb{CP}^{1|1}$ has global top holomorphic form, but does not admit flat Ricci tensor.

3. Fermionic complex dimension two and higher

We will see that the vanishing of the scalar curvature of the bosonic submanifold from flat super-Ricci tensor is restricted to complex fermionic dimension one. First let us study the case of fermionic complex dimension two. Let the expansion of the Kahler potential be[1]

$$K(z,\bar{z},\theta,\bar{\theta}) = G(z,\bar{z}) + \sum_{a\bar{b}} F_{a\bar{b}}\theta^a\bar{\theta}^{\bar{b}} + H(z,\bar{z})\theta^4\,. \tag{3.1}$$

Then the supermetric has the following block form

$$\mathcal{G} = \begin{pmatrix} A_{i\bar{j}} & B_{i\bar{\theta}^{\bar{b}}} \\ C_{\theta^a\bar{j}} & D_{\theta^a\bar{\theta}^{\bar{b}}} \end{pmatrix}, \tag{3.2}$$

[1]Henceforth we will use the notation θ^4 for $\theta^1\bar{\theta}^1\theta^2\bar{\theta}^2$.

where[2]

$$A_{i\bar{j}} = G_{,i\bar{j}} + \sum_{a\bar{b}} F_{a\bar{b},i\bar{j}}(z,\bar{z})\theta^a\bar{\theta}^{\bar{b}} + H_{,i\bar{j}}(z,\bar{z})\theta^1\bar{\theta}^1\theta^2\bar{\theta}^2 \,, \tag{3.3}$$

$$B_{i\bar{\theta}^{\bar{b}}} = \sum_{a} F_{a\bar{b},i}(z,\bar{z})\theta^a + H_{,i}(z,\bar{z})\hat{\theta^4}^{\bar{b}} \,, \tag{3.4}$$

$$C_{\theta^a\bar{j}} = \sum_{\bar{b}} F_{a\bar{b},\bar{j}}(z,\bar{z})\bar{\theta}^{\bar{b}} + H_{,\bar{j}}(z,\bar{z})\hat{\theta^4}^{a} \,, \tag{3.5}$$

$$D_{\theta^a\bar{\theta}^{\bar{b}}} = F_{a\bar{b}}(z,\bar{z}) + H(z,\bar{z})\hat{\theta^4}^{a\bar{b}} \,, \tag{3.6}$$

$$= F_{a\bar{b}}(z,\bar{z}) + H(z,\bar{z})\epsilon_{a\bar{b}c\bar{d}}\theta^c\bar{\theta}^{\bar{d}} \,. \tag{3.7}$$

Here the notation $\hat{\theta^4}^{\bar{b}}$ means removing the $\bar{\theta}^{\bar{b}}$ from $\theta^4 \equiv \theta^1\bar{\theta}^1\theta^2\bar{\theta}^2$, and likewise for $\hat{\theta^4}^{a}$ and $\hat{\theta^4}^{a\bar{b}}$. In the following, we let $g_{i\bar{j}} \equiv \partial_i\bar{\partial}_{\bar{j}}G$ denote the Kähler metric on the purely bosonic submanifold of the supermanifold.

First we compute the fermionic expansion of the determinant of $A_{i\bar{j}}$. Utilizing the general expansion formula

$$\begin{aligned} \det(I+\Gamma) &= \exp[\mathrm{tr}\ln(I+\Gamma)] \\ &= 1 + \sum_a \Gamma_{aa} + \frac{1}{2}\sum_{a,b}(\Gamma_{aa}\Gamma_{bb} - \Gamma_{ab}\Gamma_{ba}) + \cdots \end{aligned} \tag{3.8}$$

it is straightforward to find

$$\det A = \det(g_{i\bar{j}})\left[1 + \mathrm{tr}_{i\bar{j}}\,\tilde{F}_{a\bar{b}}\theta^a\bar{\theta}^{\bar{b}} + \left(\mathrm{tr}_{i\bar{j}}\,\tilde{H} + \det_{a\bar{b}}\mathrm{tr}_{i\bar{j}}\,\tilde{F} - \mathrm{tr}_{i\bar{j}}\det_{a\bar{b}}\tilde{F}\right)\theta^4\right]. \tag{3.9}$$

The notations used here needs some explanations. The double derivatives of $F_{a\bar{b}}$ can be regarded as a tensor of matrices, with $a\bar{b}$ and $i\bar{j}$ as two pairs of matrix indices. $g_{i\bar{j}} \equiv \partial_i\bar{\partial}_{\bar{j}}G$ is the metric on the bosonic submanifold, and we use its inverse to raise the indices on $i\bar{j}$. Then we define the following simplified notations

$$(\tilde{F}_{a\bar{b}})^{\bar{i}}_{\bar{j}} \equiv g^{\bar{i}k}F_{a\bar{b},k\bar{j}}(z,\bar{z}) \,, \tag{3.10}$$

$$\tilde{H}^{\bar{i}}_{\bar{j}} \equiv g^{\bar{i}k}H_{,k\bar{j}}(z,\bar{z}) \,, \tag{3.11}$$

where $\tilde{H}^{\bar{i}}_{\bar{j}}$ is a matrix in $\bar{i}\bar{j}$ indices, and $(\tilde{F}_{a\bar{b}})^{\bar{i}}_{\bar{j}}$ is a double matrix. There is difference in the order of taking matrix operations in the two pairs of indices, and we have used the following notations

$$\mathrm{tr}_{i\bar{j}}\det_{a\bar{b}}\tilde{F} \equiv \frac{1}{2}\sum_{a\bar{b}c\bar{d}}\epsilon_{a\bar{b}c\bar{d}}\tilde{F}^{\bar{i}}_{a\bar{b},\bar{j}}\tilde{F}^{\bar{i}}_{c\bar{d},\bar{j}} \,, \tag{3.12}$$

$$\det_{a\bar{b}}\mathrm{tr}_{i\bar{j}}\,\tilde{F} \equiv \frac{1}{2}\sum_{a\bar{b}c\bar{d}}\epsilon_{a\bar{b}c\bar{d}}\left(\sum_{\bar{i}}\tilde{F}^{\bar{i}}_{a\bar{b},\bar{i}}\right)\left(\sum_{\bar{j}}\tilde{F}^{\bar{j}}_{c\bar{d},\bar{j}}\right). \tag{3.13}$$

[2]The convention for the holomorphic superderivative is from the left and the anti-holomorphic derivative is from the right. In particular, there is no minus sign in B.

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For example, $\mathrm{tr}_{i\bar{j}} \det_{a\bar{b}} \tilde{F}$ means that we first regard $\tilde{H}^{\bar{i}}_{\bar{j}}$ as a matrix labelled by $a\bar{b}$ with each matrix element a matrix labelled by $(\bar{i},\bar{j})$ itself, and take the determinant in $a\bar{b}$ using the matrix multiplication in $(\bar{i},\bar{j})$ when we multiplying the matrix elements. The result determinant is a matrix itself in $(\bar{i},\bar{j})$ and one can take the trace over it. Similar explanation holds for $\det_{a\bar{b}} \mathrm{tr}_{i\bar{j}} \tilde{F}$.

To compute $\det^{-1}(D - CA^{-1}B)$, one first has

$$A^{-1} = \left(1 - \tilde{F}_{a\bar{b}}\theta^a\bar{\theta}^{\bar{b}} - \tilde{H}\theta^4 + 2\left(\det_{a\bar{b}} \tilde{F}\right)^{\bar{i}}_{\bar{j}}\theta^4\right) \cdot g^{-1}\,, \tag{3.14}$$

then the expansion $CA^{-1}B$ follows

$$\begin{aligned}(CA^{-1}B)_{a\bar{b}} &\equiv C_{\theta^a\bar{i}}(A^{-1})^{\bar{i}j}B_{j\bar{\theta}^{\bar{b}}} \\ &= -\sum_{c\bar{d}} F_{a\bar{d},\bar{i}}g^{\bar{i}j}F_{c\bar{b},j}\theta^c\bar{\theta}^{\bar{d}} - \Big(F_{a\bar{d},\bar{i}}g^{\bar{i}j}H_{,j} + H_{,\bar{i}}g^{\bar{i}j}F_{a\bar{b},j} - \\ &\qquad - \epsilon^{c\bar{d}e\bar{f}}F_{a\bar{d},\bar{i}}(\tilde{F}_{e\bar{f}})^{\bar{i}}_{\bar{k}}g^{\bar{k}j}F_{c\bar{b},j}\Big)\theta^4\,. \end{aligned} \tag{3.15}$$

The inverse determinant, using the following expansion formula,

$$\begin{aligned}\det^{-1}(I-\Gamma) &= \exp[-\,\mathrm{tr}\ln(I-\Gamma)] \\ &= 1 + \sum_a \Gamma_{aa} + \frac{1}{2}\sum_{a,b}(\Gamma_{aa}\Gamma_{bb} + \Gamma_{ab}\Gamma_{ba}) + \cdots \end{aligned} \tag{3.16}$$

becomes

$$\begin{aligned}\det^{-1}(D - CA^{-1}B) = (\det F_{a\bar{b}})^{-1}\Big\{&1 - F^{\bar{a}b}\left(H\epsilon_{b\bar{a}c\bar{d}} + F_{b\bar{d},\bar{i}}g^{\bar{i}j}F_{c\bar{a},j}\right)\theta^c\bar{\theta}^{\bar{d}} + \\ &+\Big[-g^{\bar{i}j}\,\mathrm{tr}\left(F^{-1}\partial_{\bar{i}}F\right)\partial_j H - g^{\bar{i}j}\,\mathrm{tr}\left(F^{-1}\partial_j F\right)\partial_{\bar{i}}H + \\ &\quad + \epsilon^{a\bar{b}c\bar{d}}\left(g^{\bar{j}k}\partial_k F F^{-1}\partial_{\bar{i}}F\right)_{a\bar{b}}\left(g^{\bar{i}l}\partial_l\partial_{\bar{j}}F\right)_{c\bar{d}} + \\ &\quad + Hg^{\bar{i}j}\,\mathrm{tr}^{\bar{a}}_{\bar{b}}\left(F^{-1}\partial_{\bar{i}}FF^{-1}\partial_j F\right) + \\ &\quad + Hg^{\bar{i}j}\,\mathrm{tr}\left(F^{-1}\partial_{\bar{i}}F\right)\mathrm{tr}\left(F^{-1}\partial_j F\right) + \\ &\quad + \det_{a\bar{b}}\mathrm{tr}_{i\bar{j}}\left(\partial_j FF^{-1}\partial_{\bar{i}}F\right) - \\ &\quad - \mathrm{tr}_{i\bar{j}}\det_{a\bar{b}}\left(g^{\bar{i}k}\partial_k FF^{-1}\partial_{\bar{j}}F\right)\Big]\theta^4\Big\}. \end{aligned} \tag{3.17}$$

Finally, putting everything together, we obtain the superdeterminant of the super metric $\mathcal{G}$

$$\begin{aligned}\mathrm{sdet}\,\mathcal{G} = \frac{\det g}{\det F}\Big\{&1 + \left(g^{\bar{j}i}F_{a\bar{b},i\bar{j}} - HF^{\bar{d}c}\epsilon_{c\bar{d}a\bar{b}} - F^{\bar{d}c}F_{c\bar{b},\bar{j}}g^{\bar{j}i}F_{a\bar{d},i}\right)\theta^a\bar{\theta}^{\bar{b}} + \\ &+\Big[\mathrm{tr}\,\tilde{H} + \det_{a\bar{b}}(\mathrm{tr}_{i\bar{j}}\,\tilde{F}) - \mathrm{tr}_{i\bar{j}}(\det_{a\bar{b}}\,\tilde{F}) + \\ &\quad + g^{\bar{i}j}\,\mathrm{tr}(F^{-1}\partial_{\bar{i}}\partial_j F)H - \epsilon^{a\bar{b}c\bar{d}}(\mathrm{tr}_{i\bar{j}}\,\tilde{F}_{a\bar{b}})\left(g^{\bar{i}j}\partial_j FF^{-1}\partial_{\bar{i}}F\right)_{c\bar{d}} - \\ &\quad - g^{\bar{i}j}\,\mathrm{tr}(F^{-1}\partial_{\bar{i}}F)\partial_j H - g^{\bar{i}j}\,\mathrm{tr}(F^{-1}\partial_j F)\partial_{\bar{i}}H + \\ &\quad + \epsilon^{a\bar{b}c\bar{d}}\left(g^{\bar{j}k}\partial_k FF^{-1}\partial_{\bar{i}}F\right)_{a\bar{b}}\left(g^{\bar{i}l}\partial_l\partial_{\bar{j}}F\right)_{c\bar{d}} + \end{aligned}$$

$$+ Hg^{\bar{i}j}\,\mathrm{tr}^{\bar{a}}_{\bar{b}}(F^{-1}\partial_{\bar{i}}FF^{-1}\partial_j F) + Hg^{\bar{i}j}\,\mathrm{tr}(F^{-1}\partial_{\bar{i}}F)\,\mathrm{tr}(F^{-1}\partial_j F) +$$
$$+ \det_{a\bar{b}}\mathrm{tr}_{i\bar{j}}(\partial_j FF^{-1}\partial_{\bar{i}}F) - \mathrm{tr}_{i\bar{j}}\det_{a\bar{b}}\left(g^{\bar{i}k}\partial_k FF^{-1}\partial_{\bar{j}}F\right)\Big]\theta^4\Big\}. \tag{3.18}$$

The condition for Ricci-flatness, $\mathrm{sdet}\,\mathcal{G} = 1$, implies the following set of equations,

$$\det g = \det F, \tag{3.19}$$
$$0 = -HF^{\bar{d}c}\epsilon_{c\bar{d}a\bar{b}} + g^{\bar{j}i}(F_{a\bar{b},i\bar{j}} - F_{a\bar{d},i}F^{\bar{d}c}F_{c\bar{b},\bar{j}})\,, \tag{3.20}$$
$$\begin{aligned} 0 = {} & \mathrm{tr}\,\tilde{H} + \det_{a\bar{b}}(\mathrm{tr}_{i\bar{j}}\tilde{F}) - \mathrm{tr}_{i\bar{j}}(\det_{a\bar{b}}\tilde{F}) + \\ & + Hg^{\bar{i}j}\,\mathrm{tr}(F^{-1}\partial_{\bar{i}}\partial_j F) - \epsilon^{a\bar{b}c\bar{d}}\,\mathrm{tr}_{i\bar{j}}\tilde{F}_{a\bar{b}}(g^{\bar{i}j}\partial_j FF^{-1}\partial_{\bar{i}}F)_{c\bar{d}} - \\ & - g^{\bar{i}j}\,\mathrm{tr}(F^{-1}\partial_{\bar{i}}F)\partial_j H - g^{\bar{i}j}\,\mathrm{tr}(F^{-1}\partial_j F)\partial_{\bar{i}}H + \\ & + \epsilon^{a\bar{b}c\bar{d}}(g^{\bar{j}k}\partial_k FF^{-1}\partial_{\bar{i}}F)_{a\bar{b}}(g^{\bar{i}l}\partial_l\partial_{\bar{j}}F)_{c\bar{d}} + \\ & + Hg^{\bar{i}j}\,\mathrm{tr}^{\bar{a}}_{\bar{b}}(F^{-1}\partial_{\bar{i}}FF^{-1}\partial_j F) + Hg^{\bar{i}j}\,\mathrm{tr}(F^{-1}\partial_{\bar{i}}F)\,\mathrm{tr}(F^{-1}\partial_j F) + \\ & + \det_{a\bar{b}}\mathrm{tr}_{i\bar{j}}(\partial_j FF^{-1}\partial_{\bar{i}}F) - \mathrm{tr}_{i\bar{j}}\det_{a\bar{b}}(g^{\bar{i}k}\partial_k FF^{-1}\partial_{\bar{j}}F)\,. \end{aligned} \tag{3.21}$$

To figure out the geometrical meaning of these equations, it is best to regard these as matrix equations. Multiplying the second equation by $g^{\bar{b}a}$ and taking trace,

$$g^{\bar{j}i}[\mathrm{tr}\,F^{-1}\partial_i\partial_{\bar{j}}F - \mathrm{tr}(F^{-1}\partial_i FF^{-1}\partial_{bj}F)] - 2H\det F^{-1} = 0\,. \tag{3.22}$$

It is easy to verify the following equality for matrix F

$$\mathrm{tr}\,\partial_i\partial_{\bar{j}}(\ln F) = \mathrm{tr}(F^{-1}\partial_i\partial_{\bar{j}}F - F^{-1}\partial_i FF^{-1}\partial_{\bar{j}}F)\,, \tag{3.23}$$

from which one has

$$-2H\det{}^{-1}F = -g^{\bar{j}i}\partial_i\partial_{\bar{j}}(\mathrm{tr}\ln F)\,. \tag{3.24}$$

Because $\det F = \det g$, the righthand side of the equation is exactly the Ricci scalar curvature $R \equiv -g^{\bar{j}i}\partial_i\partial_{\bar{j}}\,\mathrm{tr}\ln g$ of the bosonic base manifold. So one finally finds out the scalar curvature for the bosonic submanifold

$$R = -2H\det{}^{-1}g\,. \tag{3.25}$$

g is the metric of the bosonic manifold which is certainly non-degenerate, $\det g \neq 0$. So as long as $H \neq 0$, the bosonic base manifold need not have vanishing scalar curvature.

H is the coefficient of the quartic term in the fermionic expansion, which naturally disappears if there is only one complex fermionic dimension as studied in [4]. It is exactly the finite cutoff determined by the dimension of the fermionic degrees of freedom in the fermionic expansion causes the anomaly in the dimension one.

When the complex fermionic dimension is larger than two, one has similar result concerning the scalar curvature for the bosonic submanifold. The Kähler potential has expansion in fermionic coordinates

$$K(z,\bar{z},\theta,\bar{\theta}) = G(z,\bar{z}) + \sum_{a\bar{b}} F_{a\bar{b}}\theta^a\bar{\theta}^{\bar{b}} + \sum_{a\bar{b}c\bar{d}} H_{a\bar{b}c\bar{d}}(z,\bar{z})\theta^a\bar{\theta}^{\bar{b}}\theta^c\bar{\theta}^{\bar{d}} + \cdots\,. \tag{3.26}$$

where the scalar H in the complex fermionic dimension two has been replaced by a totally anti-symmetric tensor $H_{a\bar{b}c\bar{d}}$. Up to $\theta\bar{\theta}$ order in the superdeterminant expansion there are similar equations

$$\det g = \det F \,, \tag{3.27}$$

$$F^{\bar{d}c} H_{c\bar{d}a\bar{b}} = g^{\bar{j}i} \left(F_{a\bar{b},i\bar{j}} - F_{a\bar{d},i} F^{\bar{d}c} F_{c\bar{b},\bar{j}} \right) . \tag{3.28}$$

The scalar curvature of the bosonic manifold is then

$$R = -F^{\bar{d}c} F^{\bar{b}a} H_{c\bar{d}a\bar{b}} \,. \tag{3.29}$$

It is obvious that the scalar curvature can generally take any value. This proves our claim.

However, one can not conclude that Calabi-Yau theorem holds for general supermanifold when the fermionic complex dimension is two or higher yet. The reason is that there are equations from the higher order expansions of the superdeterminant, and one may wonder if they impose topological or geometrical conditions on the bosonic submanifold. We will study the case for fermionic dimension two and find some hints.

Use the first two equations to simplify the last equation, and after a bit of algebra, we find

$$\mathrm{tr}_{i\bar{j}} \det{}_{a\bar{b}} \left(F g^{\bar{i}k} \partial_k \partial_{\bar{j}} \ln F \right) = H \left(g^{\bar{i}j} \partial_{\bar{i}} \ln R \partial_j \ln R + H g^{\bar{i}j} \partial_{\bar{i}} \partial_j \ln H + \frac{R}{2} \right) . \tag{3.30}$$

Notice that using the previous two equations and the expression for H, the righthand side of the equation is determined by the metric of the bosonic submanifold alone. Now after solving for H, one has five equations on $F_{a\bar{b}}$ in terms of $g_{i\bar{j}}$, one from (3.19), three from (3.20) after using (3.25) and one from (3.21). Naively this should lead to one consistency condition for $g_{i\bar{j}}$, however notice that some of these are nonlinear second order differential equations, and such simple counting may be wrong.

One can look at some simple examples. For example, the supermanifold $\mathbb{CP}^{1|2}$ is super-Ricci flat. Actually the super-Ricci flat metric is the Fubini-Study metric, which is from the Kähler potential

$$K(z, \bar{z}, \theta^a, \bar{\theta}^{\bar{b}}) = \ln \left(1 + z\bar{z} + \theta^1 \bar{\theta}^{\bar{1}} + \theta^2 \bar{\theta}^{\bar{2}} \right) . \tag{3.31}$$

as one can easily verify.

If one regards $F_{a\bar{b}}$ and their first order derivatives as independent functions, these five equations become nonlinear first order equations and there will be more independent functions than constraints. Then there is no consistency condition and so no constraints on the bosonic submanifold, and these equations merely states that $F_{a\bar{b}}$ is correlated to $g_{i\bar{j}}$ for super Ricci flat metric. Although we have no solid proof, we believe that the Calabi-Yau theorem does hold for fermionic complex dimension two or higher. Certainly it will be interesting to either prove or disprove this conjecture in the future.

Acknowledgments

The author would like to thank Jeremy Michelson and Xinkai Wu for helpful discussions. This work is supported by NSF grant PHY-0244811 and DOE grant DE-FG01-00ER40899.

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